COMPLETENESS OF COUNTABLE INDUCTIVE LIMITS OF BANACH SPACES

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In this survey on the question of the completeness of (LB) spaces we will focus our attention on the two following problems:

(P1) Is the completion of an (LB) space again an (LB) space?

(P2) Is a regular (LB) space complete?

As we will see, an affirmative answer to (P2) impplies an affirmative answer to (P1) but neither (P1) nor (P2) are presently proved to be true or false.

We start by fixing some definitions. A Hausdorff locally convex space $[E,\tau]$ is an (LB) space if there exists an strictly increasing sequence $\{[E_n,\tau_n]\}_{n\in\mathbb{N}}$ of Banach spaces such that $\tau_{n+1}|_{E_n} \leq \tau_n$ for all $n\in\mathbb{N}$, $E = \bigcup_n E_n$ and τ is the strongest Hausdorff locally convex topology in E making continuous all the injections $E_n \longrightarrow E$ (i.e. τ is the inductive locally convex Hausdorff topology induced by the τ_n's). We therefore write

$$[E,\tau] = \operatorname*{ind}_{n\to} [E_n,\tau_n] \quad \text{or} \quad E = \operatorname*{ind}_{n\to} E_n \tag{1}$$

Every (LB) space is a (DF) space (= "dual Fréchet") and we will often use freely the good stability properties of these latter spaces.

If each E_n is only a Fréchet space, E is called an (LF) space. (LF) spaces will arise occasionally in our discussion. The (LB) or (LF) space (1) is called regular (resp. boundedly retractive) if each bounded set $B \subset E$ is contained and bounded in some step E_n (resp. if furthermore $\tau|_B = \tau_n|_B$). Boundedly retractive (LB) spaces are not of much interest in our setting because they are always complete (in fact they are quasi-complete (DF) spaces). On the other hand regular (LB) spaces are easily recognized by means of the following criterion: If B_n is the closed unit ball of the Banach space E_n , $n \in \mathbb{N}$, the (LB) space $E = \operatorname*{ind}_{n\to} E_n$ is regular whenever B_n is closed in E for all $n \in \mathbb{N}$. Indeed if B is a bounded set in E , there exists $n \in \mathbb{N}$ and $\lambda > 0$ such that $B \subset \overline{\lambda B_n}^E$ because E is a (DF) space. It follows that $B \subset \lambda B_n$, B_n being closed in E .

An essential tool in studying (LF) spaces is the

Grothendieck Factorization Theorem

If F is a Fréchet space, $E = \operatorname{ind}_{n\to} E_n$ is an (LF) space and $f: F \longrightarrow E$ is a linear continuous map, then there exists $n \in \mathbb{N}$ such that $f(F) \subset E_n$ and $f: F \longrightarrow E_n$ is continuous (i.e. f factorizes continuously through E_n).

This Factorization Theorem provides two immediate consequences:

a) No (LF) space is a Fréchet space.

b) The (LF) space E is regular iff E is locally complete (= for any closed bounded disk $B \subset E$, the associated normed space E_B is a Banach space)

THE ORIGIN OF THE PROBLEM

In 1954 Grothendieck constructs for the first time an (LF) space which is metrizable ([6]). Afterwards the class of metrizable (LF) spaces turned out to be rather large (every non-normable Fréchet space, and every Fréchet space with unconditional basis possesses a dense (LF) subspace). If E is such a metrizable (LF) space, the completion $\hat{E}$ of E is not an (LF) space, by a). Even the class of (ultra-)bornological spaces (i.e. general inductive limits of (Banach) normed spaces) does not enjoy good properties against completion. One can see, for instance, the counterexample of Kōmura-Kōmura [8], using the continuum hypothesis, and the counterexample of Valdivia [12], exhibiting, without the continuum hypothesis, a bornological space which is a dense hyperplane of his non-bornological completion. We next describe in more detail a further example, introducing the space $\mathscr{C}(X)_c$ of scalar-valued continuous functions on a completely regular topological space X (endowed with the compact-open topology), and using the following

THEOREM OF NACHBIN-SHIROTA

A completely regular topological space X is real-compact iff $\mathscr{C}(X)_c$ is bornological.

Associated to every completely regular topological space X we denote by $X_{\mathbb{R}}$ the set X endowed with the strongest completely regular topology having the same compact sets than the original. It is known that $\mathscr{C}(X_{\mathbb{R}})_c$ is the completion of $\mathscr{C}(X)_c$. If $X = X_{\mathbb{R}}$ (homeomorphic), X is called a $k_{\mathbb{R}}$-space. Then, as a consequence of the Nachbin-Shirota Theorem, we have

<u>Example 1</u>

Let X be a real compact topological space such that $X_{\mathbb{R}}$ is not real-compact. Then $\mathscr{C}(X)_c$ is a bornological space such that its completion $\mathscr{C}(X_{\mathbb{R}})_c$ is not bornological. A good deal of such topological spaces X appears in [2].

SOME POSITIVE RESULTS

It is worth noting that no (LB) space is metrizable and the above considerations about metrizable (LF) spaces do not apply to the class of (LB) spaces. Consequently the starting problems (P1) and (P2) stand open and we come back to them proving that, in fact, they are related. If (P2) has an affirmative answer then (P1) has an affirmative answer:

<u>Proposition 1</u>

If there exists an (LB) space $E = \operatorname{ind}_{n\to} E_n$ such that its completion $\hat{E}$ is not an (LB) space then there exists a non complete regular (LB) space.

<u>Proof</u>

If B_n is the closed unit ball of E_n, then $C_n := \overline{nB_n}^{\hat{E}}$, $n \in \mathbb{N}$, is a fundamental system of bounded sets in $\hat{E}$ by the known properties of (DF) spaces. Therefore the bornological space $\hat{E}^{bor} = \operatorname{ind}_{n\to} \hat{E}_{C_n}$ associated to $\hat{E}$ is an <u>(LB) space</u>. It is also locally complete ($\Leftrightarrow$ <u>regular</u>, by b)) because $\hat{E}$ and $\hat{E}^{bor}$ have the same bounded sets. We finally note that $\hat{E}^{bor}$ <u>is not complete</u>. Otherwise the continuous injection $E \longrightarrow \hat{E}^{bor}$ would extend continuously to a continuous identity $\hat{E} \longrightarrow \hat{E}^{bor}$ and then the topological identity $\hat{E} = \hat{E}^{bor}$ conflicts with the fact that $\hat{E}$ is not an (LB) space.

We will study now two instances in which the problem (P1) has an affirmative answer.

THEOREM 1 (J. Mujica) (Compact Neighborhhods)

Let $E = \operatorname{ind}_{n\to} E_n$ be an (LB) space. If there exists a Hausdorff locally convex topology τ on E such that the closed unit ball B_n of each E_n is τ-compact, then E is complete.

Application of the Theorem of Mujica: The space $H(K)$ of all germs of holomorphic functions on the compact set K of a complex Fréchet space is complete.

We recall that $H(K)$ is defined as the (LB) space $\operatorname{ind}_{n\to} \mathcal{H}^{\infty}(U_n)$ where $\{U_n\}_{n\in\mathbb{N}}$ is a countable decreasing basis of open neighborhoods of K in E and $\mathcal{H}^{\infty}(U_n)$ is the Banach space of bounded holomorphic functions in U_n endowed with the sup. norm. The problem of completeness of the space $H(K)$ of germs of holomorphic functions in K was finally solved by S. Dineen [4] using a rather cumbersome technique. However the Mujica's Theorem (see [10]) supplies a more simple proof and more general methods using only functional analytic techniques.

Next we study a second partial answer of (P1) in which the following notations are in order: $[E,\tau] = \operatorname{ind}_{n\to} E_n$ is an (LB) space, $[\hat{E},\hat{\tau}]$ is the completion of $[E,\tau]$, B_n is the closed unit ball of the Banach space E_n (the sequence $\{B_n\}$ can be assumed increasing) and $C_n := \bar{B}_n^{\hat{E}}$ for all $n \in \mathbb{N}$. We define the (LB) space $[F,\Phi] := \operatorname{ind}_{n\to} \hat{E}_{C_n}$ and we consider the identity

$$I : [F,\Phi] \longrightarrow [\hat{E},\hat{\tau}] \qquad (2)$$

which is continuous (note that $[F,\Phi]$ is nothing but the bornological space $\hat{E}^{bor}$ associated to $\hat{E}$ as in the proof of the Proposition 1).

THEOREM 2 (J. M. García-Lafuente) (Complete Neighborhoods)

With the above notations, t.f.a.e.:

a) $[\hat{E},\hat{\tau}]$ is an (LB) space

b) $[\hat{E},\hat{\tau}]$ is topologically isomorphic to $[F,\Phi]$

c) C_n is Φ-complete for every $n \in \mathbb{N}$

d) Φ is a complete topology.

Proof:

b) $\Rightarrow$ a) and b) $\Rightarrow$ c) are trivial and a) $\Rightarrow$ b) is a consequence of the Köthe-Grothendieck Closed Graph Theorem for (LF) spaces in view of the continuity of (2).

d) $\Rightarrow$ b) The inclusion $[E,\tau] \longrightarrow [F,\Phi]$ is continuous because each map $E_n \longrightarrow [F,\Phi]$ is bounded. So, if Φ is complete, there exists a continuous extension $J : [\hat{E},\hat{\tau}] \longrightarrow [F,\Phi]$. This fact toghether with (2) proves b).

c) $\Rightarrow$ d) Since $F = \bigcup_n \hat{E}_{C_n} = \bigcup_n nC_n$, the (increasing) sequence $\{nC_n\}_n$ is absorbing in the (σ-barrelled) space $[F,\tau]$. By the De Wilde-Houet Theorem ([3], Theorem 2), we have

$$\overline{\bigcup_n nC_n} \subset \text{alg. clo.} \bigcup_n \overline{nC_n} \qquad \text{(closures in } [\hat{F},\hat{\Phi}])$$

Therefore taking any $\alpha > 0$ one has

$$\hat{F} = \bar{F} = \overline{\bigcup_n nC_n} \subset (1+\alpha) \bigcup_n \overline{nC_n} = \text{(by b))} = (1+\alpha) \bigcup_n nC_n = F .$$

Note that if for every sequence $\{\varepsilon_n\}$ of strictly positive real numbers, the Φ-neighborhood $\bigcup_n \varepsilon_n C_n$ is $\hat{\tau}$-closed (each C_n is) then the condition d) of the above theorem fulfills by the Closed Neighborhood Completeness Theorem ([7], 3.2.4.).

We will finally exhibit several examples supporting (as the Theorems 1 and 2 do) an affirmative answer to (P1) and (P2). No counterexample has been found so far.

<u>Example 2 (The celebrated incomplete coechelon Köthe space)</u>

Let $a^k := (a^k(n))_{n\in\mathbb{N}}$, $k \in \mathbb{N}$, be a sequence of echelons with $0 < a^k(n) < a^{k+1}(n)$ for all $k,n \in \mathbb{N}$. We define the c_o-weighted sequence space

$$c_o(1/a^k) := \{ x = (x(n))_n \in \mathbb{R}^{\mathbb{N}} ;\ \lim_{n\to\infty} \frac{x(n)}{a^k(n)} = 0 \}$$

endowed with its usual weighted topology of Banach space, and then we consider the (LB) space $E := \operatorname{ind}_{k\to} c_o(1/a^k)$. For a suitable choosing of the echelons a^k, E turns out to be neither complete nor regular ([9], 31.6.) but its completion $\hat{E}$ is an (LB) space (see [11], Example 8.8.9.).

Example 3

Let $\mathbb{K}^{(\mathbb{N})}$ be the countable dimensional linear space endowed with the strongest locally convex topology. With E as in the above example, $F := E \times \mathbb{K}^{(\mathbb{N})}$ is an incomplete (LB) space whose completion is the (LB) space $\hat{E} \times \mathbb{K}^{(\mathbb{N})}$. This example supplies an (LB) space in which each step defining the limit is not dense in the limit F, while in the Example 2, each step is dense in the limit E ((LB) spaces of type 1 and 2 respectively in terminology of Saxon-Narayanaswami)

Example 4 ((LB) spaces of Moscatelli type)

Let $[L, \| \ \|]$ be a normal Banach sequence space, and let $\{Y_k\}_{k\in\mathbb{N}}$ and $\{X_k\}_{k\in\mathbb{N}}$ be two sequences of Banach spaces such that for all $k \in \mathbb{N}$ Y_k is a proper subspace of X_k and $\|x\|_{X_k} \leq \|x\|_{Y_k}$ for all $x \in Y_k$ (in particular one has continuous embeddings). For each $n \in \mathbb{N}$ we define the Banach space

$$E_n := \{ (x_k)_{k\in\mathbb{N}} \in \prod_{k<n} X_k \times \prod_{k\geq n} Y_k \quad ; \ ((\|x_k\|_{X_k})_{k<n} , (\|x_k\|_{Y_k})_{k\geq n}) \in L \}$$

endowed with the obvious norm coming from L. The (LB) space $E := \underset{n\to}{\operatorname{ind}} E_n$ is called an (LB) space of Moscatelli type and for these kind of spaces we have the following positive answer to (P2):

Theorem (Moscatelli)

Every regular (LB) space of Moscatelli type is complete.

Example 5 (Weighted inductive limits of continuous function spaces)

Let X be a locally compact topological space and let $\mathcal{V} := \{v_n\}_{n\in\mathbb{N}}$ a decreasing sequence of strictly positive continuous functions defined on X. For each $n \in \mathbb{N}$ we define the "weighted" space

$$C(v_n)(X) := \{ f:X \longrightarrow \mathbb{R} \ ; \ f \text{ is continuous and } \|f\|_n := \sup_{t\in X} v_n(t)|f(t)| < +\infty \}$$

endowed with the complete norm-topology of $\| \ \|_n$, and we consider its closed subspace

$$C(v_n)_o(X) := \{ f:X \longrightarrow \mathbb{R} \ ; \ f \text{ is continuous and } v_n \circ f \text{ vanishes at infinity} \}$$

wich is as well a Banach space under the relative topology. We then define the weighted inductive limits

$$\mathcal{V}C(X) := \operatorname{ind}_{n\to} C(v_n)(X)$$

$$\mathcal{V}_o C(X) := \operatorname{ind}_{n\to} C(v_n)_o(X)$$

and we ask about conditions on the sequence $\mathcal{V}$ in order to have good properties of completeness of these (LB) spaces.

1) $\mathcal{V} = \{v_n\}$ is said to have the property (V) if for every $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that v_m/v_n vanish at infinity.

2) $\mathcal{V} = \{v_n\}$ is said "regularly decreasing" if for each $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that for every $Y \subset X$

$$\inf_{y\in Y} \frac{v_m(y)}{v_n(y)} > 0 \quad \text{impplies} \quad \inf_{y\in Y} \frac{v_k(y)}{v_n(y)} > 0 \quad \text{for all} \quad k > m$$

It is then known ([1], Proposition 20 and Theorem 7) the following

Theorem 3

Let X be a locally compact topological space and $\mathcal{V}$ a decreasing sequence of strictly positive continuous functions on X. Then

a) If $\mathcal{V}$ satisfies the property (V) then $\mathcal{V}C(X) = \mathcal{V}_o C(X)$ (topologically) and this common (LB) space is boundedly retractive and hence complete.

b) If $\mathcal{V}$ is regularly decreasing then the (LB) spaces $\mathcal{V}_o C(X)$ and $\mathcal{V}C(X)$ are complete .

We will finally mention that recently has been proved ([5]) that the Theorem 3 a) remains valid when X is a paracompact $k_{\mathbb{R}}$-space (for example any metric space) without any other compactness hypothesis.

REFERENCES

[1] *Bierstedt, K. D.,* An introduction to locally convex inductive limits. Functional Analysis and its applications. Lecture Notes I.C.P.A.M., Nice (1986), 35-133

[2] *Blasco, J. L.,* On μ-spaces and $k_{\mathbb{R}}$-spaces. Proc. Amer. Math. Soc. 67 (1977), 179-186

[3] *De Wilde, M., Houet, C.,* On increasing sequences of absolutely convex sets in locally convex spaces. Math. Ann. 192 (1971), 257-261

[4] *Dineen, S.,* Holomorphic germs on compact subsets of locally convex spaces. Functional Analysis Holomorphy and Approximation Theory. L. N. in M. 843, Springer, Berlin (1981), 247-263

[5] *García-Lafuente, J. M.,* Weighted inductive limits of continuous functions spaces with non-locally compact domain (preprint)

[6] *Grothendieck, A.,* Sur les espaces (F) and (DF). Summa Brasil Math. 3 (1954), 57-123

[7] *Jarchow, H.,* Locally Convex Spaces. B. G. Teubner, Stuttgart, 1981

[8] *Kōmura, T., Kōmura, Y.,* Sur les espaces parfaits de suites et lurs gnralisations. J. Math. Soc. Japan 15 (1963), 319-338

[9] *Köthe, G.,* Topological Vector Spaces I . Springer, Berlin, 1969

[10] *Mujica, J.,* A completeness criterion for inductive limits of Banach spaces. Functional Analysis, Holomorphy and Approximation Theory. North Holland (1984), 319-329

[11] *Pérez-Carreras, P., Bonet, J.,* Barrelled Locally Convex Spaces. North-Holland Math. Studies, 131, Amsterdam, 1987

[12] *Valdivia, M.,* On the completion of bornological spaces. Archiv d. Math. (1977), 608-613.

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